Analog Computation Using Single-Electron Circuits

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Abstract. Analog computation is a processing method that solves a given problem by utilizing an analogy of a physical system to the problem. An idea is presented here for relating the behavior of single-electron circuits to analog computation. As an instance, a method is proposed for solving a combinatorial problem, the three-colorability problem, by using the properties of single-electron circuits. In problem solving, a single-electron circuit is constructed that is analogous to a given problem; then, through an annealing procedure, the circuit is made to settle down to its minimun energy state. The correct solution to the problem can be obtained by checking the final arrangement of electrons in the circuit. Analog computation is a promising architecture for single-electron computing systems.

1. Introduction

One of the promising areas of research in microelectronics is the development of novel computing systems based on single-carrier electronics. To create such systems, we must employ a method of computation that makes the best use of the properties of single-electron circuits. This paper proposes one such computation method: that is, analog computation utilizing the energy-minimizing principle in single-electron circuits.

Single-carrier electronics is a technology for manipulating electronic functions by controlling the transport of individual electrons, through the use of single-electron circuits [1,2]. It has been receiving increasing attention because it affords the possibility of producing computation systems that provide novel functions beyond those of conventional devices. To take steps toward this goal, various logic devices consisting of single-electron circuits have been proposed that perform digital processing in the manner of Boolean operation [see 3-6 for examples].

In this article, we present for future discussion an idea of a novel computation device based on non-Boolean operation. It is a single-electron analog computation device. Analog computation is a way of processing that solves a given problem by applying an analogy of a physical system to the problem. By relating the properties of single-electron circuits with the method of analog computation, we will be able to create a novel computation device that furnishes quick solutions to combinatorial problems.

In the following sections, first, the concept of analog computation is explained using a known example of an analog-computation system, a soap-film system that solves the Steiner tree problem (Section 2). After that we present the idea for relating single-electron circuits to analog computation. This concept is illustrated with an example, a solution to the three-colorability problem, that utilizes the energy-minimizing behavior of single-electron circuits. The construction of the circuits for solving the problem is presented (Section 4). Results of computer-simulated operation of the circuit is then discussed to demonstrate the problem-solving behavior (Section 5). The authors hope that this will stimulate the thinking of readers who are aiming to develop new processing devices that utilize single-electron phenomena.

2. The Concept of Analog Computation

2.1. What is Analog Computation?

Analog computation is a way of processing that solves a mathematical problem by applying an analogy of a physical system to the problem. To solve the problem in this way, you prepare an appropriate physical system and represent each problem variable by a physical quantity in the system. If the mathematical relations between the physical quantities are analogous to those of the problem, then you can find the solution to the problem by observing the behavior of the system and measuring the corresponding physical quantities. A way of processing based on this principle is called analog computation.

The analog computation is quite different from the commonly used binary-digital computation. In the digital approach, you first devise an algorithm (a set of instructions for finding the solution to a problem), then execute each step of the algorithm in the manner of Boolean operation. In contrast, analog computation is concerned with no symbolic Boolean operation; instead it utilizes the properties of a physical system to perform the mathematical operations required for the solution. An important feature of analog computation is concurrency or parallelism in computing, through which analog computation can provide the possibility of solving complex problems in a short time.

2.2. An Example of Analog Computation: Solving the Steiner Tree Problem by Means of a Soap-Film System

Consider the following problem (Fig. 1). Connect n points on a plane with a graph of minimum overall length, using additional junction points if

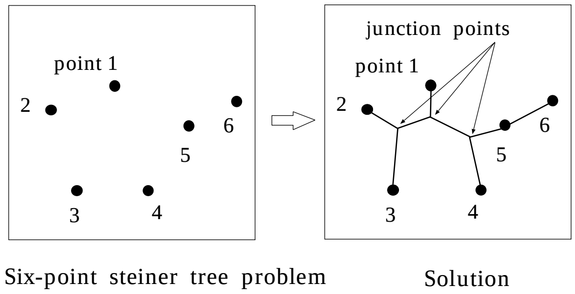

Fig. 1. The Steiner tree problem. Connect given points on a plane with a graph of minimum overall length. This is difficult to solve using existing computers because it requires enormous computing time.

necessary. This is a combinatorial problem called the *Steiner tree problem.* Plainly expressed, the problem is"to connect n cities by a road network of minimum total length."

This problem is intractable for digital computation. There are many possible graphs with junction points, and we must examine all the possible ones to find the minimum solution. The number of computational steps required increases exponentially with the number n of original points. Indeed, the Steiner tree problem belongs to the class of NP-hard problems (non-deterministic polynomial-time hard problems). Except for inefficient exponential-time procedures, no algorithm is known for the solution. This problem therefore requires enormous computing time to solve and is virtually unsolvable for large values of n.

Neverthless, there is an ingenious analog-computation method that can quickly solve the problem (see [7]). We use soap films to make a physical system analogous to the problem (Fig. 2). Prepare two parallel glass plates and insert n pins between the plates to represent the points; then dip the structure into a soap solution and withdraw it. The soap film will connect the n pins in the minimum Steiner-tree graph. The computing process is parallel and instantaneous, so we can obtain the solution in a very short time regardless of the number n of the pins.

In this analog computation, the energy-minimizing principle is well utilized for problem solving. Any physical system changes its configuration to decrease its total energy. In liquids at rest, the relevant energy components are the gravitational potential energy and the surface energy.

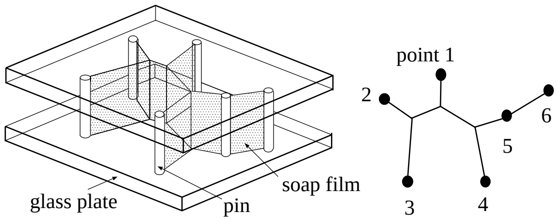

Fig. 2. A soap film solution to the Steiner tree problem. The problem can be quickly solved by utilizing the equilibrium of a soap-film system (see [7]).

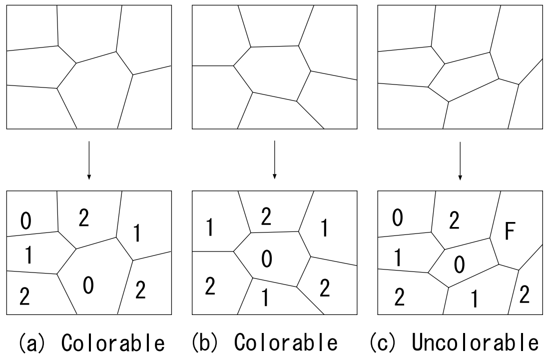

Fig. 3. Coloring of a given map with three colors. (a), (b) Colorable maps. The numbers 0, 1, and 2 represent three colors. (c) An uncolorable map. The trial solution fails on reaching region F.

The latter is dominant in a thin soap film, and so a soap-film system changes its configuration to minimize its total area (therefore its length) and thereby its surface energy.

Strictly speaking, it is not possible to be certain, in this system, that the absolute minimum solution can always be obtained. Depending on the angle at which the system is withdrawn from the soap solution, the soap-film network sometimes assumes topologies different from the optimum one that gives the minimum network length (this is due to the fact that, in a soap film, many local minima exist in the energy-topology relation). Even in such cases, however, the networks obtained are always nearly equal to the minimum one. Hence it can be said that the system works well in general.

3. Single-Electron Circuit for Solving the Colorability Problem

3.1. Relating Single-Electron Circuits to Combinatorial Problems

It is interesting to speculate what analog computations are possible using the properties of single-electron devices. We here utilize the property of the single-electron circuit changing its state to decrease its free energy. This can be used for solving combinatorial problems, as in the soap film computation above. By constructing a single-electron circuit such that its free energy function is related to the objective function of a given combinatorial problem, we will be able to solve the problem simply by observing to which state the circuit will settle down.

Setting aside the issue of the feasibility of fabricating actual devices under existing process technologies, the authors present here an instance of a single-electron circuit system that can be applied to combinatorial problems. It is a circuit system for solving the *three-colorability problem.* In the following, we will describe the three-colorability problem, and then will propose the structure of the single-electron circuit system for solving this problem.

3.2. The Three-Colorability Problem

Consider the following problem: can the countries on a given map be colored with *three* colors such that no two countries that share a border have the same color (Fig. 3)? This is called the three-colorability problem and is difficult to solve for a map with many countries. There are colorable maps and uncolorable ones, but we cannot tell whether a given map can be colored before examining all the possible colorings. (The problem is easy if we can use four colors because it has been proved that four colors suffice for any map.) The three-colorability problem belongs to the class of NP-complete (nondeterministic polynomial-time complete problems), and is intractable for digital computation because only exponential-time algorithms are known for the solution.

This problem is reduced to graph coloring (Fig. 5). Any map can be converted into a corresponding dual graph by reducing each country to a vertex and by drawing an edge between two vertices if the corresponding two countries share a border. Coloring the map is then equivalent to coloring the

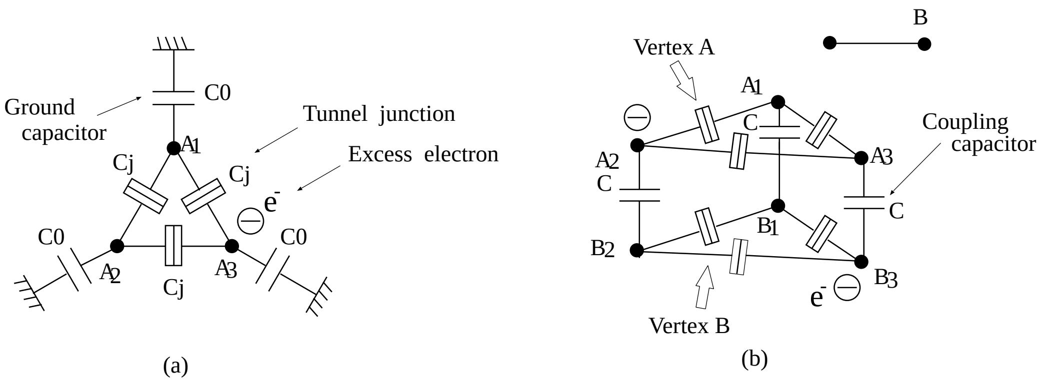

Fig. 4. Construction of a single-electron circuit analogous to the three-colorability problem. (a) A triangular subcircuit representing a vertex of the graph. (b) A circuit analogous to the two connected vertices A and B (for simplicity, the ground capacitance for each node is omitted in the illustration).

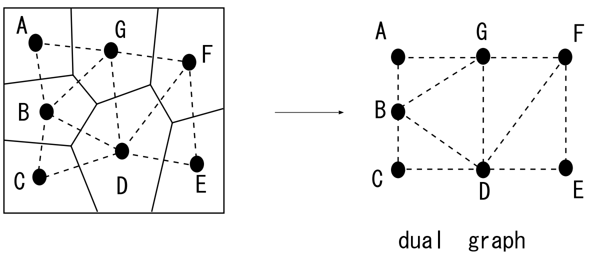

Fig. 5. The dual graph for the map in Fig. 3(a). Each vertex is colored in one of three colors. Two vertices connected by an edge cannot have the same color.

graph, under the rule that two vertices connected by an edge cannot have the same color.

3.3. Single-Electron Circuit for Problem Solving

The following describes a way of solving the three-colorability problem by using the single-electron circuit. Our work is first to construct a single-electron circuit analogous to a given map for the problem and then to solve the problem by using the circuit.

A. Implementing a Dual Graph by Using a Single-Electron Circuit. Taking the map given in Fig. 3(a) as an example, we construct the analogous single-electron circuit for problem solving. The map can be converted into the dual graph shown in Fig. 5, reducing our task to constructing a single-electron circuit analogous to the graph.

To represent a vertex on the graph, we use a triangular subcircuit illustrated in Fig. 4(a), which consists of three identical tunnel junctions (C_j) connected in series to form a ring with three nodes A1, A2,A3. One *excess* electron (e^-) is put in the subcircuit, and it occupies one of the three nodes. A ground capacitance C_0 exists between each node and ground. We define that the three nodes represent three differing colors (e.g., $A1$ represents red, $A2$ blue, and $A3$ green), and that the color of the vertex is equal to the color of the node occupied by the excess electron (e.g., the vertex is colored green if the electron is on node $A3$). Hereafter, we call this subcircuit with the excess electron a *triangle subcircuit* and call the excess electron simply an electron.

We first implement two vertices, A and B, that are connected by an edge. This is done by coupling two triangle subcircuits in the manner illustrated in Fig. 4(b), using a coupling capacitor C to connect each two nodes that represent the same color. (A ground capacitance exists for each node, but it is not illustrated for simplicity.) Free energy in this coupling circuit is equal to electrostatic energy and takes a large value for a state in which two electrons occupy same-color nodes to face each other (e.g., occupying nodes $A1$ and $B1$); therefore, to keep its energy level at a minimum, the circuit will tend to avoid such *single-color* states. In consequence, two electrons in the two coupled triangle subcircuits will occupy two nodes that represent differing colors; e.g., if the

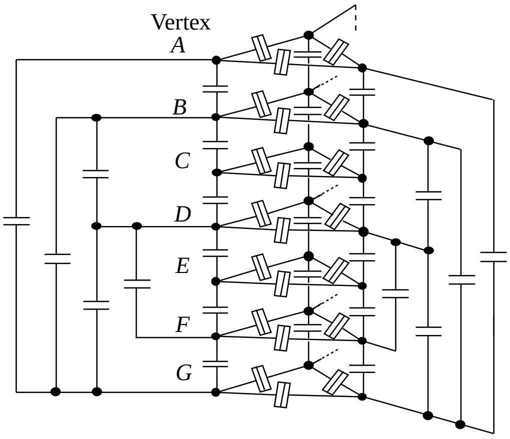

Fig. 6. The analogous circuit for solving the three-colorability problem for the graph in Fig. 5 (or for the map in Fig. 3(a)).

electron in triangle subcircuit A is on node $A1$, then the electron in triangle subcircuit B will be on a node of a differing color, $B2$ or $B3$.

A complete circuit analogous to the graph given in Fig. 5 can be obtained by connecting seven triangul subcircuits, using 33 coupling capacitors C, as depicted in Fig. 6. (Each node has a ground capacitance, but these are omitted in the illustration.) Electrons in two neighboring triangle subcircuits tend to occupy nodes of *differing color* to reduce the total energy of the circuit, which satisfies the requirement of the three-colorability problem.

It should be stressed that this procedure of constructing analogous circuits can be applied to every other map. For any problem map given, we can construct the corresponding analogous circuit by combining identical triangle subcircuits and coupling capacitors.

B. Solving the Problem by Using the Constructed Circuit The three-colorability problem asks whether a given map is colorable, and the answer is either "yes" or "no". To solve the problem by using the analogous circuit, we carry out the following procedure. Put the circuit in an initial state (any state will do), then let the circuit settle down to its equilibrium state with the minimum electrostatic energy, and then check to see whether two electrons in any coupled triangle subcircuits are on nodes of *differing* colors. If they are, the answer is "yes" and the colors of the occupied nodes indicate the a way of coloring in which the map can be colored. If they are not, the answer is "no". (As for the circuit in Fig. 6, we will obtain a "yes" answer because the circuit is for the leftmost colorable map in Fig. 3.) This solution is based on the following two principles:

(a) Electrostatic energy in the analogous circuit has a large value when two electrons face each other at neighboring nodes of the same color. Consequently the circuit will change its state to minimize the number of such electron pairs and, if possible, to reduce such pairs to zero.

(b) "A map is colorable" is equivalent to "in the analogous circuit, at least one arrangement of electrons exists such that *no* two electrons face each other at same-color nodes." (Let's call such a state of electron arrangement a *satisfaction state.*) In contrast, a circuit for an uncolorable map has no such satisfaction state.

In the minimum-energy state, the circuit for a colorable map is in a satisfaction state, and we will find that electron pairs in any coupled triangle subcircuits are on dots of *differing* colors. In a circuit for an *un*colorable map, no satisfaction state can be attained, so we will find one or more electron pairs occupying the dots of the *same* color.

A similar solution using single-electron circuits should exist for other NP-complete problems. This is because every NP-complete problem belongs to the same class and one can be converted into another.

4. Simulating Circuit Operation of Problem Solving

For problem solving, it is essential that, starting with a given initial state, analogous circuits should settle down to their minimum-energy states. Unfortunately, analogous circuits in general have many states of locally minimum energy, as will be shown later, and in consequence we cannot be certain, as things stand, that the circuit can achieve the state of globally minimum energy without becoming stuck in the local minima. To make the circuit converge exactly to the minimum energy state, we here use the *annealing method* to operate the analogous circuits successfully. Using this

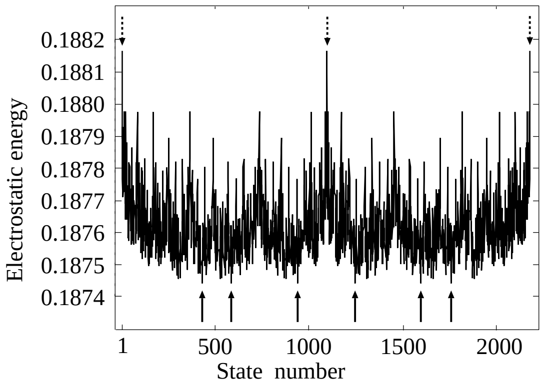

Fig. 7. Electrostatic energy vs. electron arrangement for the circuit given in Fig. 6.

method, we can obtain the global minimum state in the circuit and thence the correct solution to the problem. The details are described in the following sections, using results of computer simulations.

4.1. Energy Function and Local Minima in Analogous Circuits

The electrostatic energy of single-electron circuits is a function of the electron arrangement in the circuit. For various instances of the three-colorability problem, we designed the corresponding analogous circuits and calculated the energy of each circuit for all possible arrangements of electrons. We found that all the circuits have many local minima in their energy functions.

To illustrate this situation, a configuration of the energy function is depicted in Fig. 7, taking the circuit shown in Fig. 6 as an example. The horizontal axis in the figure indicates the number of the electron arrangement. One number corresponds to one arrangement of electrons; 2187 arrangements are possible because three possible arrangements exist for an electron in each of the seven triangle circuits. (In calculation, the circuit parameters were assumed as: tunnel junction capacitance $C_j = 100$ aF, coupling capacitance $C = 100$ aF, and ground capacitance C_0 on each circuit node = 1 aF. There is no special reason for these values – any other value can be used.)

The energy of this circuit becomes minimum for several specific electron arrangements (the arrangements of numbers 429, 584, 939, 1249, 1604, and 1759; indicated by solid arrows in the figure), which correspond to the satisfaction states representing the correct solution to the graph (or the map) coloring. But it can also be seen that many local minima exist that have energy values close to that of the minimum-energy states. It is therefore not possible to be certain that the circuit can always achieve the correct solution without getting stuck in the local minima. (The electron arrangements of numbers 1, 1094, and 2187, indicated by dashed arrows, correspond to states of monochromatic coloring – i.e., coloring the graph (or the map) with a single color. These states have the maximum energy value.)

To elucidate the effect of the local minima, we will here observe by computer simulation the state transition in analogous circuits. In simulation, we used a Monte Carlo method that was combined with the basic equations for electric-charge distribution, charging energy, and tunneling probability; the probabilistic characteristic of electron tunneling was introduced through the use of random numbers (see Ref. 8). The co-tunneling phenomenon was ignored for simplicity, and temperature was assumed to be 0 K.

The result is illustrated in Fig. 8, for the sample circuit shown in Fig. 6. The circuit was initially set at the state of monochromatic coloring, then was left changing its state without restraint. After some transition time the circuit stabilized in a final state. This procedure, a *trial*, was repeated many times using a different series of random numbers; the results of three trials are illustrated in the figure. We observed that the circuit in most cases became stuck in a local minimum and could not reach the global-minimum energy state. Very rarely, the circuit successfully stabilized in the global minimum, but this was by sheer chance.

4.2. Annealing Operation Method

To overcome the above difficulty, we consider operating the circuit by the annealing method. This method consists of the following four steps (Fig. 9).

(1) Put the analogous circuit into a heat bath, and set the circuit at an initial state (any state will do).

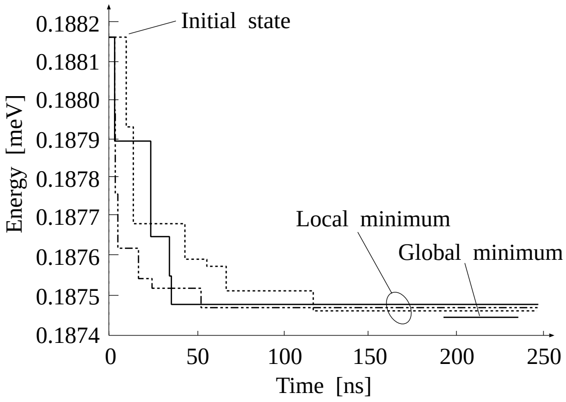

Fig. 8. State transition in the analogous circuit given in Fig. 6 (computer simulation). The results of three trials are plotted. The circuit finally became stuck in a local-minimum state.

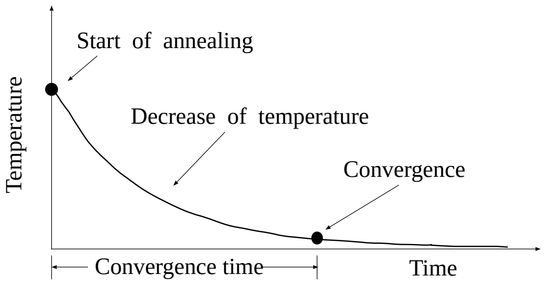

Fig. 9. Concept of the annealing procedure.

(2) Initially increase the temperature of the heat bath to a maximum value at which the circuit changes its state or electron arrangement randomly.
(3) Carefully decrease the temperature of the heat bath until the circuit arranges its electrons in an equilibrium state (or until the circuit reaches convergence).
(4) Check the final arrangement of electrons in the circuit to see whether the circuit is in the satisfaction state.

If the lowering of the temperature is done slowly enough, the analogous circuit can reach thermal equilibrium at each temperature, and so can approach the global-minimum state with decrease in temperature. And thus we can obtain the solution to the problem by observing the final state of the circuit.

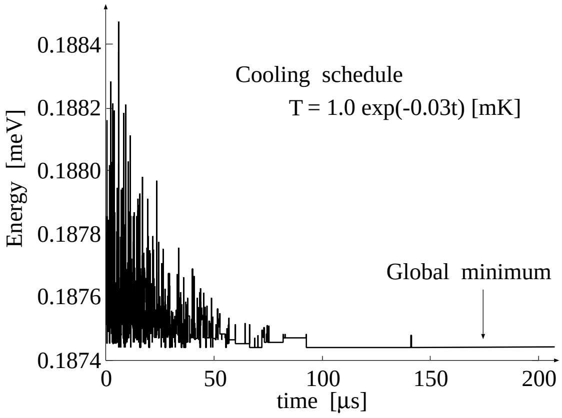

Fig. 10. State transition in the analogous circuit with annealing (computer simulation). The result for a trial is shown. The circuit can successfully reach the global-minimum state.

The term "annealing" is principally used for the metallurgical process for obtaining a perfect crystal without deformations and dislocations – heating a body of metal to near its melting point and then cooling it slowly to room temperature. In this study, we adopted annealing as an effective method for achieving successful operation of analogous circuits.

4.3. Convergence to the Minimum Energy State through the Annealing

We simulated the process of annealing operation for various analogous circuits and confirmed successful convergence to the minimum energy state. For the *cooling schedule* (a decrement function for lowering the temperature in annealing), we used the *natural cooling* given by $T = T_0 \exp(-\rho t)$, where T is the temperature, T_0 is an initial value of the temperature, ρ is a cooling-speed coefficient, and t is time. (Parameters T_0 and ρ govern the convergence of a circuit during annealing. The values for successful convergence can be inferred by experience from the size of a given analogous circuit.)

The simulation result for annealing is illustrated in Fig. 10, taking the circuit of Fig. 6 as an example. The circuit was initially set at a monochromatic-coloring state, then was left changing its state under the natural cooling given by $T_0 = 1.0$ mK and $\rho = 0.03$ (mK/s). The result for a trial is plotted in the figure. It is shown that the circuit successfully reaches the global-minimum energy state. (At the first stage of annealing, the circuit was excited transitorily, as can be seen in Fig. 10, to states the energy of which is higher than that for monochromatic coloring, 0.18816 meV. These "upper" states are the states in which, owing to thermal excitation, a conduction electron of a node metal is extracted from the node and is transferred to another node. This situation vanishes with the decrease in annealing temperature and has no influence on problem solving.)

In this way, we can find the global minimum state of analogous circuits and thence the correct solution to given problems.

5. Conclusion

Analog computation is a processing method that solves mathematical problems by applying an analogy of a physical system to the problem. An idea for relating single-electron circuits with analog computation was presented. As an instance, a method was proposed for solving the three-colorability problem by using the properties of single-electron circuits. In problem solving, we construct a single-electron circuit analogous to the problem and find the minimum energy state of the circuit through annealing. By checking the final arrangement of electrons in the circuit, we can obtain the correct solution to the given problem. Analog computation is a promising architecture for single-electron processing systems.

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